

LOG KODAIRA DIMENSION OF HOMOGENEOUS VARIETIES

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ABSTRACT. Let V be a complex algebraic variety, homogeneous under the action of a complex algebraic group. We show that the log Kodaira dimension of V is non-negative if and only if V is a semi-abelian variety.

1. INTRODUCTION

Throughout this short note, we work over the field of complex numbers. Given a smooth quasi-projective variety V , we recall the definition of the log Kodaira dimension $\bar{\kappa}(V)$. Take any projective variety X which contains V as a dense open subset. By blowing up subvarieties of the (reduced) boundary $D := X \setminus V$, we may assume that X is smooth and D is a simple normal crossing divisor. In this case, the pair (X, D) is called a *smooth projective compactification of V* . Let K_X be a canonical divisor of X . The *log Kodaira dimension* $\bar{\kappa} = \bar{\kappa}(X, K_X + D)$ is defined as the unique value

$$\bar{\kappa} \in \{-\infty, 0, 1, \dots, \dim V\}$$

such that for some positive constants α, β , we have

$$\alpha m^{\bar{\kappa}} \leq h^0(X, m(K_X + D)) \leq \beta m^{\bar{\kappa}}$$

for any sufficiently large and divisible positive integer m . As $\bar{\kappa}$ is independent of the choice of the smooth projective compactification (X, D) (cf. [4, §11.1]), we may set $\bar{\kappa} := \bar{\kappa}(V)$.

Also, recall that a Cartier divisor D on a projective variety X is *pseudo-effective* if its numerical equivalence class is contained in the closure of the convex cone spanned by the effective divisor classes on X .

We may now state the main result of this note:

Theorem 1.1. *Let X be a smooth projective variety, and G a connected algebraic group (possibly nonlinear) of automorphisms of X . Assume that G has a dense open orbit V in X , and $D := X \setminus V$ is a simple normal crossing divisor. Then the following conditions are equivalent.*

- (1) *The log canonical divisor $K_X + D$ is pseudo-effective.*

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- (2) $K_X + D$ is linearly equivalent to 0.
- (3) The log Kodaira dimension $\bar{\kappa}(V)$ is non-negative.
- (4) G is a semi-abelian variety.

We recall that a *semi-abelian variety* is an algebraic group G that lies in an exact sequence

$$1 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where T is an algebraic torus (isomorphic to a product of copies of the multiplicative group \mathbb{G}_m), and A an abelian variety; then G is connected and commutative.

Here are a few words about the proof of Theorem 1.1, which is very simple. If V is not a semi-abelian variety, we show that the general member of some covering family of affine lines on G is not contractible to a point in V . Chevalley's structure theorem for algebraic groups is also used, so our proof is quite geometric.

From Theorem 1.1, we derive the following purely group-theoretic statement:

Corollary 1.2. *Let G be a connected algebraic group, and $H \leq G$ a closed subgroup. Then $\bar{\kappa}(G/H) \leq 0$, with equality if and only if H contains a closed normal subgroup N of G such that G/N is a semi-abelian variety.*

2. PROOFS OF THEOREM 1.1 AND COROLLARY 1.2

The following result is well-known. See [5, Lemma 5.11] for a generalization to singular pairs.

Lemma 2.1. *Let Y be a smooth projective variety and A a simple normal crossing divisor on Y . Suppose that C is the general member of a covering family of rational curves on Y . Let d be the number of times that C meets A , i.e., the degree of the (reduced) support of the divisor $\nu^*(A|_C)$, where $\nu : \mathbb{P}^1 \rightarrow C$ is the normalization. Then we have:*

- (1) If $d \leq 2$, then $(K_Y + A) \cdot C \leq 0$.
- (2) If $d \leq 1$, then $(K_Y + A) \cdot C < 0$. In particular $K_Y + A$ is not pseudo-effective.

We now prove Theorem 1.1.

(2) \Rightarrow (3) and (3) \Rightarrow (1) follow readily from the definitions of the log Kodaira dimension and pseudo-effectivity.

(4) \Rightarrow (2). Since the semi-abelian variety $G \subseteq \text{Aut}(X)$ is commutative and acts faithfully on X , the stabilizer of any point of V is trivial, and hence $V \cong G$. Also, the action of the Lie algebra of G on X yields a trivialization of the log tangent bundle $T_X(-\log D)$ (see [8, Main Thm.] and [2, Thm. 2.5.1]). In particular, the determinant of $T_X(-\log D)$ is trivial as well, i.e., $K_X + D \sim 0$.

(1) \Rightarrow (4). Fix a point $x_0 \in V$ and denote by $H \leq G$ its isotropy subgroup, so that V is G -equivariantly isomorphic to the homogeneous space G/H , with both the G -actions from the left. Since G acts faithfully on V , we have:

Claim 2.2. *H contains no non-trivial closed normal subgroup of G .*

Next we show:

Claim 2.3. *H contains the image of every additive one-parameter subgroup of G .*

We prove Claim 2.3. Assume the contrary that there exists an additive one-parameter subgroup of G , i.e., a homomorphism of algebraic groups $u : \mathbb{G}_a \rightarrow G$ (where \mathbb{G}_a denotes the additive group), with image not contained in H . Then the \mathbb{G}_a -orbit $u(\mathbb{G}_a) \cdot x_0$ is a curve and isomorphic to \mathbb{G}_a/F , where the isotropy subgroup F is a finite subgroup of \mathbb{G}_a and hence is trivial. Thus we obtain an embedding

$$f : \mathbb{A}^1 \cong \mathbb{G}_a \cong u(\mathbb{G}_a) \cdot x_0 \hookrightarrow G/H,$$

$$t \longmapsto u(t) \cdot x$$

and hence a rational curve $C \subset X$ which intersects the boundary D at at most one point. Also, the translates of C by G form a covering family of rational curves on X . By Lemma 2.1, it follows that $K_X + D$ is not pseudo-effective, contradicting our assumption. This proves Claim 2.3.

We return to the proof of Theorem 1.1. By Chevalley's structure theorem (see [7, Thm. 16]), G lies in a unique extension

$$1 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where L is a connected linear algebraic group, and A an abelian variety. Consider the unipotent radical $R_u(L)$; this is a closed connected normal subgroup of G . If $R_u(L)$ is non-trivial, consider the last non-trivial term of its lower central series, U . Then U is a closed connected normal subgroup of G , isomorphic to the additive group of a finite-dimensional vector space. Hence, U is not contained in H by Claim 2.2; but then G has an additive one-parameter subgroup with image not contained in H , contradicting Claim 2.3. So $R_u(L)$ is trivial, i.e., L is reductive. Thus, the derived subgroup $[L, L]$ is semi-simple, and hence generated by images of additive one-parameter subgroups. So $[L, L] \subseteq H$ by Claim 2.3. As $[L, L]$ is a normal subgroup of G , Claim 2.2 implies that $[L, L]$ is trivial. So L is an algebraic torus. Hence G is a semi-abelian variety.

This completes the proof of Theorem 1.1.

Next we prove Corollary 1.2.

Denote by N the kernel of the action of G on G/H , i.e., the largest (closed) normal subgroup of G contained in H . Replacing G, H with $G/N, H/N$, respectively, we may assume that G acts faithfully on G/H .

By [3, Thm. 2], there exists a projective compactification X on G/H such that the natural G -action on G/H from the left extends to an algebraic action on X . Using equivariant resolution of singularities (see [6]), we may assume that X is smooth and $X \setminus V$ is a simple normal crossing divisor. Then the desired assertion follows from Theorem 1.1. This proves Corollary 1.2.

3. CONCLUDING REMARKS

3.1. With the assumptions of Theorem 1.1, the action of the Lie algebra of G on X preserves D , and hence yields a linear map

$$\mathrm{op}_{X,D} : \mathrm{Lie}(G) \longrightarrow H^0(X, T_X(-\log D)),$$

where $T_X(-\log D)$ denotes the log tangent bundle. Moreover, the pull-back of $T_X(-\log D)$ to V is just the tangent bundle T_V , which is generated by the image of $\mathrm{op}_{X,D}$. It follows that the induced map

$$\wedge^n \mathrm{Lie}(G) \longrightarrow H^0(X, -(K_X + D))$$

is nonzero, where $n := \dim(V) = \dim(X)$. In particular, $-(K_X + D)$ is effective. Thus, the pseudo-effectivity of $K_X + D$ in Theorem 1.1 (1) immediately implies Theorem 1.1 (2): $K_X + D \sim 0$.

3.2. Theorem 1.1 is applicable to an arbitrary (not necessarily smooth) equivariant compactification $Y \supseteq V$ as long as Y is normal, the boundary $E := Y \setminus V$ is a divisor, and $K_Y + E$ is \mathbb{Q} -Cartier. Indeed, by equivariant resolution of singularities, there exists a G -equivariant birational morphism $f : X \rightarrow Y$ such that X is smooth, projective and equipped with a G -action, and $D := X \setminus V$ (which is the inverse of E) is a simple normal crossing divisor. By the logarithmic ramification divisor formula, we have

$$K_X + D = f^*(K_Y + E) + R_f$$

where R_f is an effective f -exceptional divisor. Thus the pairs (X, D) and (Y, E) (as compactifications of V) have the same log Kodaira dimensions: $\kappa(X, K_X + D) = \kappa(Y, K_Y + E)$, and the log canonical divisor $K_X + D$ is pseudo-effective if and only if so is $K_Y + E$.

In particular, if $K_Y + E$ is pseudo-effective, then so is $K_X + D$, and hence both $V \subseteq X$ and $V \subseteq Y$ are the embeddings of the semi-abelian variety V in X and Y , respectively.

3.3. The normal projective equivariant compactifications of a given semi-abelian variety are well-understood by work of Alexeev (see [1]), where they are called semi-abelic varieties. In particular, for a semi-abelian variety G given by an extension

$$1 \rightarrow T \rightarrow G \rightarrow A \rightarrow 1$$

where T is an algebraic torus and A an abelian variety, and a normal projective equivariant compactification Y of G , the morphism $q : G \rightarrow A$ extends to a T -invariant morphism

$$f : Y \longrightarrow A,$$

which is a fibration in projective toric varieties with the big torus T . Thus, f is the Albanese map of the variety Y .

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